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COMMON FIXED POINT THEOREMS IN HILBERT SPACE

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ABSTRACT

The aim of this present paper is to obtain a common fixed point for continuous self mappings on Hilbert space. Our purpose here is to generalize the our previous result [3]

KEYWORDS: Common Fixed Point, Hilbert Space, Continuous Self Mappings

2010 Mathematics Subject Classification: 47H10.

INTRODUCTION

Kannan [3] proved that a self mapping T on complete metric space (X,d) satisfying the condition

$$d(x,y) \le \alpha\{ d(x,Tx)+d(y,Ty)\} \text{ for all } x,y \text{ in X where } 0<\alpha<\frac{1}{2}$$ has a unique fixed in (X,d).

Koparde and Waghmode [5] have proved fixed point theorem for a self mappings T on a closed subset S of Hilbert space H, satisfying the Kannan type condition

$$\|Tx-Ty\|^2 \le \alpha\left\{\|x-Tx\|^2+\|y-Ty\|^2\right\} \quad \text{for all } x,y \text{ in } S \text{ and } x\ne y\ ; 0\le a<\tfrac{1}{2}$$

Koparde and Waghmode [5] also extended this result to the pair of mappings T_1 and T_2 and to their power p,q are some positive integers.

In this paper we have obtained a unique fixed point for self mapping satisfying rational inequality

$$\|Tx-Ty\| \le \alpha\left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}\right\}+\beta\|x-y\| \text{ for all } x,y \text{ in } S \text{ and } x\ne y;\ \frac{1}{2}>\alpha\ge 0,\ \beta\ge 0 \text{ and } 2\alpha+\beta<1.$$

In a Hilbert space

MAIN RESULTS

We proved followingg fixed point theorems.

Theorem 1: Let S be a closed subset of a Hilbert space H. Let T be a self mappings on S satisfying the following condition

$$\|Tx-Ty\| \le \alpha\left\{\frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}\right\}+\beta\|x-y\| \text{ for all } x,y \text{ in } S \text{ and } x\ne y;\ \frac{1}{2}>\alpha\ge 0,\ \beta\ge 0 \text{ and } 2\alpha+\beta<1.$$

Then T has a unique common fixed point.

Proof: Let S be a closed subset of a Hilbert space H . Let T be a self mappings on S. Let $x_0 \in S$ be any arbitrary point in S.

Define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by

$x_{n+1} = Tx_n = T^{n+1}x_0$, for $n = 0,1,2, ...$

Suppose that $x_{n+1} \neq x_n$ for $n = 0,1,2, ...$

For any integer $n \geq 1$.

$$\|x_{n+1} - x_n\| = \|Tx_n - Tx_{n-1}\|$$

$$\leq \alpha \left\{ \frac{\|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_{n-1}\|^2}{\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\|} \right\} + \beta \|x_n - x_{n-1}\|$$

$$= \alpha \frac{\|x_n - x_{n+1}\|^2 + \|x_{n-1} - x_n\|^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} + \beta \|x_n - x_{n-1}\|$$

$$\leq \alpha \frac{\{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|\}^2}{\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|} + \beta \|x_n - x_{n-1}\|$$

$$\leq \alpha \|x_n - x_{n+1}\| + \alpha \|x_{n-1} - x_n\| + \beta \|x_n - x_{n-1}\|$$

$$= \alpha \|x_{n+1} - x_n\| + \alpha \|x_n - x_{n-1}\| + \beta \|x_n - x_{n-1}\|$$

i.e. $\|x_{n+1} - x_n\| \leq \alpha \|x_{n+1} - x_n\| + \alpha \|x_n - x_{n-1}\| + \beta \|x_n - x_{n-1}\|$

$$\Rightarrow (1-\alpha)\|x_{n+1} - x_n\| \leq (\alpha + \beta)\|x_n - x_{n-1}\|$$

$$\Rightarrow \quad \|x_{n+1} - x_n\| \leq \frac{\alpha + \beta}{1-\alpha}\|x_n - x_{n-1}\|$$

If $k = \frac{\alpha + \beta}{1-\alpha}$ then $k < 1$.

$$\|x_{n+1} - x_n\| \leq k\|x_n - x_{n-1}\|$$

$$\leq k\|x_n - x_{n-1}\| \leq k^2\|x_{n-1} - x_{n-2}\| \leq k^3\|x_{n-2} - x_{n-3}\| \leq ... \leq k^n\|x_1 - x_0\|$$

i.e. $\|x_{n+1} - x_n\| \leq k^n\|x_1 - x_0\|$ for all $n \geq 1$ is integer.

Now for any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + ... + \|x_{m-1} - x_m\|$$

$$\leq k^n\|x_1 - x_0\| + k^{n+1}\|x_1 - x_0\| + ... + k^{m-1}\|x_1 - x_0\|$$

$\leq k^n \|x_1 - x_0\| \left(1 + k + ... + k^{m-n-1}\right)$

$i.e. \|x_n - x_m\| \leq \left(\dfrac{k^n}{1-k}\right) \|x_1 - x_0\| \to 0 \ as\, n \to \infty \ (k < 1)$

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since S is a closed subset of a Hilbert space H, so $\{x_n\}_{n=1}^{\infty}$ converges to a point u in S.

Now we will show that u is common fixed point of self mappings T from S into S.

Suppose that $\mathrm{T}u \neq u$.

Consider

$\|u - \mathrm{T}u\| \leq \|u - x_n\| + \|x_n - \mathrm{T}u\|$

$= \|x_n - \mathrm{T}u\|$

$= \|\mathrm{T}\, x_{n-1} - \mathrm{T}u\|$

$$\leq \alpha \left\{ \frac{\|x_{n-1} - \mathrm{T}x_{n-1}\|^2 + \|u - \mathrm{T}u\|^2}{\|x_{n-1} - \mathrm{T}x_{n-1}\| + \|u - \mathrm{T}u\|} \right\} + \beta \|x_{n-1} - u\|$$

$$\leq \alpha \left\{ \frac{\|x_{n-1} - x_n\|^2 + \|u - \mathrm{T}_m u\|^2}{\|x_{n-1} - x_n\| + \|u - \mathrm{T}_m u\|} \right\} + \beta \|x_{n-1} - u\|$$

i.e. $\|u - \mathrm{T}u\| \leq \alpha \left\{ \|x_{n-1} - x_n\| + \|u - \mathrm{T}u\| \right\} + \beta \|x_{n-1} - u\|$

$(1-\alpha)\|u - \mathrm{T}u\| \leq \alpha \|x_{n-1} - x_n\| + \beta \|x_{n-1} - u\| \quad \to 0 \quad as \quad n \to \infty$

So $\|u - \mathrm{T}u\| \leq 0.$

Hence $u = \mathrm{T}u$.

Hence u is a common fixed point of self mappings T.

Uniqueness: Suppose that there is $u \neq w$ such that $\mathrm{T}w = w$.

Consider $\| u - w \| = \| \mathrm{T}u - \mathrm{T}w\|$

$$\leq \alpha \left\{ \frac{\| u - \mathrm{T}u\|^2 + \| w - \mathrm{T}w\|^2}{\| u - \mathrm{T}u\| + \| w - \mathrm{T}w\|} \right\} + \beta \| u - w\|$$

$\leq \alpha \| u - \mathrm{T}u\| + \alpha \| w - \mathrm{T}w\| + \beta \| u - w \|$

i.e. $\| u - w \| \leq \beta \| u - w \|$

$\Rightarrow \quad \|u-w\| = 0$

Thus $u = w$.

Hence fixed point is unique.

Theorem 2: Let S be a closed subset of a Hilbert space H. Let T_1 and T_2 are self mappings on S satisfying the following condition

$$\|T_1x - T_2y\| \le \alpha\left\{\frac{\|x-T_1x\|^2 + \|y-T_2y\|^2}{\|x-T_1x\| + \|y-T_2y\|}\right\} + \beta\|x-y\| \text{ for all } x, y \text{ in } S \text{ and } x \ne y;\ \frac{1}{2} > \alpha \ge 0,\ \beta \ge 0 \text{ and } 2\alpha + \beta < 1.$$

Then T_1 and T_2 have a unique common fixed point.

Proof: Let S be a closed subset of a Hilbert space H . Let T_1 and T_2 are self mappings on S. Let $x_0 \in S$ be any arbitrary point in S.

Define a sequence $\{x_n\}_{n=1}^{\infty}$ in S by

$$\left.\begin{array}{l} x_{2n+1} = T_1 x_{2n} \\ x_{2n+2} = T_2 x_{2n+1} \end{array}\right\} \text{ for } n = 0, 1, 2, \ldots \ .$$

Suppose that $x_{n+2} \ne x_{n+1} \ne x_n$ for $n = 0, 1, 2, \ldots$.

Consider

$$\|x_1 - x_2\| = \|T_1x_0 - T_2x_1\|$$

$$\le \alpha\left\{\frac{\|x_0 - T_1x_0\|^2 + \|x_1 - T_2x_1\|^2}{\|x_0 - T_1x_0\| + \|x_1 - T_2x_1\|}\right\} + \beta\|x_0 - x_1\|$$

$$\le \alpha\,\frac{\|x_0 - x_1\|^2 + \|x_1 - x_2\|^2}{\|x_0 - x_1\| + \|x_1 - x_2\|} + \beta\|x_0 - x_1\|$$

$$\le \alpha\|x_0 - x_1\| + \alpha\|x_1 - x_2\| + \beta\|x_0 - x_1\|$$

i.e. $\quad \|x_1 - x_2\| \le \alpha\|x_0 - x_1\| + \alpha\|x_1 - x_2\| + \beta\|x_0 - x_1\|$

$$\Rightarrow (1-\alpha)\|x_1 - x_2\| \le (\alpha+\beta)\|x_0 - x_1\|$$

$$\Rightarrow \quad \|x_1 - x_2\| \le \frac{(\alpha+\beta)}{(1-\alpha)}\|x_0 - x_1\|$$

If $k = \dfrac{\alpha+\beta}{1-\alpha}$ then $k < 1$.

$$\|x_1 - x_2\| \le k\|x_0 - x_1\|$$

Now

$$\|x_2 - x_3\| = \|T_2x_1 - T_1x_2\|$$
$$= \|T_1x_2 - T_2x_1\|$$

$$\leq \alpha \left\{ \frac{\|x_2 - T_1x_2\|^2 + \|x_1 - T_2x_1\|^2}{\|x_2 - T_1x_2\| + \|x_1 - T_2x_1\|} \right\} + \beta \|x_2 - x_1\|$$

$$\leq \alpha \frac{\|x_2 - x_3\|^2 + \|x_1 - x_2\|^2}{\|x_2 - x_3\| + \|x_1 - x_2\|} + \beta \|x_2 - x_1\|$$

i.e. $\|x_2 - x_3\| \leq \alpha \|x_2 - x_3\| + \alpha \|x_1 - x_2\| + \beta \|x_2 - x_1\|$

$$\Rightarrow (1-\alpha)\|x_2 - x_3\| \leq (\alpha + \beta)\|x_1 - x_2\|$$

$$\Rightarrow \quad \|x_2 - x_3\| \leq \frac{(\alpha+\beta)}{(1-\alpha)}\|x_1 - x_2\|$$

If $k = \dfrac{\alpha+\beta}{1\text{-}\alpha}$ then $k < 1$.

$$\|x_2 - x_3\| \leq k\|x_1 - x_2\|$$
$$\leq k^2 \|x_0 - x_1\|$$

i.e. $\|x_2 - x_3\| \leq k^2 \|x_0 - x_1\|$

Hence by induction we get for integer n ≥ 1

$$\|x_n - x_{n+1}\| \leq k^n \|x_0 - x_1\|$$

Or

$$\|x_{n+1} - x_n\| \leq k^n \|x_1 - x_0\|$$

Now for any positive integer $m \geq n \geq 1$

$$\|x_n - x_m\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \ldots + \|x_{m-1} - x_m\|$$

$$\leq k^n \|x_1 - x_0\| + k^{n+1}\|x_1 - x_0\| + \ldots + k^{m-1}\|x_1 - x_0\|$$

$$\leq k^n \|x_1 - x_0\| \left(1 + k + \ldots + k^{m-n-1}\right)$$

$$i.e. \|x_n - x_m\| \leq \left(\frac{k^n}{1-k}\right)\|x_1 - x_0\| \rightarrow 0 \; as \, n \rightarrow \infty \; (k < 1)$$

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since S is a closed subset of a Hilbert space H, so $\{x_n\}_{n=1}^{\infty}$ converges to a point u in S.

Now we will show that u is common fixed point of the pair of mappings T_1 and T_2 from S into S.

First we will show that u is fixed point of T_1.

Suppose that $T_1 u \neq u$

Consider

$$\|u - T_1 u\| \leq \|u - x_{2n+2}\| + \|x_{2n+2} - T_1 u\|$$

$$\leq \|x_{2n+2} - T_1 u\|$$

$$= \|T_2 x_{2n+1} - T_1 u\|$$
$$= \|T_1 u - T_2 x_{2n+1}\|$$

$$\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|x_{2n+1} - T_2 x_{2n+1}\|^2}{\|u - T_1 u\| + \|x_{2n+1} - T_2 x_{2n+1}\|} \right\} + \beta \|u - x_{2n+1}\|$$

$$\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|x_{2n+1} - x_{2n+2}\|^2}{\|u - T_1 u\| + \|x_{2n+1} - x_{2n+2}\|} \right\} + \beta \|u - x_{2n+1}\|$$

i.e. $\|u - T_1 u\| \leq \alpha \left\{ \|u - T_1 u\| + \|x_{2n+1} - x_{2n+2}\| \right\} + \beta \|u - x_{2n+1}\|$

$(1-\alpha)\|u - T_1 u\| \leq \alpha \|x_{2n+1} - x_{2n+2}\| + \beta \|u - x_{2n+1}\| \quad \to 0$ as $n \to \infty$

So $\|u - T_1 u\| \leq 0.$

Hence $u = T_1 u$.

Similarly we can prove that u is fixed point of T_2.

Hence u is a common fixed point of mappings T_1 and T_2

Uniqueness: Suppose that there is $w \neq u$ such that $T_2 w = w$.

Consider $\| u - w \| = \| T_1 u - T_2 w \|$

$$\leq \alpha \left\{ \frac{\|u - T_1 u\|^2 + \|w - T_2 w\|^2}{\|u - T_1 u\| + \|w - T_2 w\|} \right\} + \beta \|u - w\|$$

$$\leq \alpha \| u - T_1 u\| + \alpha \| w - T_2 w\| + \beta \| u - w \|$$

i.e. $\| u - w \| \leq \beta \| u - w \|$

$\Rightarrow \| u - w\| = 0$

Thus $u = w$.

Similarly $v \in S$ if is such that $T_1 v = v$ then $u = v$

Hence u is unique common fixed point of T_1 and T_2.

Corollary

Let p and q be two positive integers such that operators T_1^p and T_2^q :S→S where S is closed subset of Hilbert space H,satisfy the inequality

$$\left\|T_1^p x - T_2^q y\right\| \le \alpha\left\{\frac{\left\|x - T_1^p x\right\|^2 + \left\|y - T_2^q y\right\|^2}{\left\|x - T_1^p x\right\| + \left\|y - T_2^q y\right\|}\right\} + \beta\left\|x - y\right\| \text{ for all } x, y \text{ in } S \text{ and } x \ne y;\ \frac{1}{2} > \alpha \ge 0,\ \beta \ge 0 \text{ and } 2\alpha + \beta < 1.$$

Then T_1 and T_2 have a unique common fixed point.

Proof: By the above theorem, the T_1^p and T_2^q have a unique common fixed point, say $u \in S$.

Since $u = T_1^p u$

therefore $T_1 u = T_1\left(T_1^p u\right)$

$= T_1^p\left(T_1 u\right)$

i.e. $T_1 u$ is also a fixed point of T_1^p .But by above theorem T_1^p has a unique fixed point.

Therefore, $T_1 u = u.$

Similarly $T_2 u = u.$

Hence u is common fixed point of T_1 and T_2.

For uniqueness ,if w is a another common fixed point of T_1 and T_2,then clearly w is a also a common fixed point of T_1^p and T_2^q

This gives w = u.

This completes the proof.

Example: Let X = [0, 1], with Euclidean metric d .Then {X, d} is a Hilbert space with the norm defined by $d(x, y) = \left\|x - y\right\|$.

$$\text{Let } \{x_n\}_{n=1}^{\infty} = \begin{cases} \dfrac{1}{2.6^{(n-1)/2}} x_0 & \text{when } n \text{ is odd} \\ \dfrac{1}{6^{n/2}} x_0 & \text{when } n \text{ is even} \end{cases}$$

be the sequence in X

and let Let T_1 and T_2 are self mappings on X defined by

$$T_1(x) = \frac{x}{3} \quad \text{and} \quad T_2(x) = \frac{x}{2}$$

Clearly T_1 and T_2 are are satisfy the rational inequality. Also sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in *X*, which is converges in *X* also it has a common point in *X*.

CONCLUSIONS

In this paper we replace Kannan inequality by rational inequality The theorems proved in this paper by using rational inequality is improved and stronger form of some earlier inequality given by Kannan [3], Koparde and Waghmode [5]. We have obtained a unique fixed point for self mapping satisfying rational inequality extended this result to the pair of mappings T_1 and T_2 and to their power p,q are some positive integers.

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